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Domain segregation in a two-dimensional system in the presence of drift

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Motivated by experiments on optical patterns we analyze two-dimensional extended bistable systems with drift after a quench above threshold. The evolution can be separated into successive stages: linear growth and diffusion, coarsening, and transport, leading finally to a quasi-one-dimensional kink-antikink state. The phenomenon is general and occurs when the bistability relates to uniform phases or two different patterns.

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In spatially extended dynamical systems the breaking of reflection symmetry along an axis due to drift has been studied, e.g., in Taylor vortices with through flow [1,2]. In convection in complex fluids such as liquid crystals, drift is produced by suitable alignment of the director at the boundaries [3]. Important features observed in the presence of drift are the transition from convective to absolute instability, and the occurrence of noise-induced structures in the convectively unstable regime.

In optical patterns drift may be generated by several factors, including oblique incidence of input light or misalignements in resonators [4–6], spatial displacement in the feedback loop of nonlinear interferometer [7,8], and angular walkoff between interacting waves in optical parametric oscillators [9]. In this context, different amounts of drift have been shown to induce transitions between different patterns, e.g., hexagons, rolls, and zig-zag [7].

We here study the influence of drift on bistable, (quasi-) two-dimensional (2D) systems quenched into the absolutely unstable region. Without drift one initially has linear growth of fluctuations, saturation, and then coarsening, where larger domains grow at the expense of smaller ones (we are not concerned here with inhomogeneous growth via a front process). Curvature of the domain walls provides the driving force in this last regime. We will show that with drift in the absolutely unstable region the system evolves to a nonperiodic array of 1D domains (stripes), whereby further coarsenig is effectively stopped. In this process a new analytic finger solution emerges, which may be relevant also in other curvature driven dynamical processes.

The investigation is motivated by experimental observations in a system made of a liquid crystal light valve (an effective Kerr medium) inserted in a ring cavity with a spatial displacement of the optical wavefront in the diffractive feedback loop. Calling L the free propagation length, and k_0 the wave number of the pump laser, the system bifurcates in the appropriate range of lateral displacements Δx from the homogeneous state toward two roll sets with wave number $q_0 \approx \sqrt{\pi k_0/L}$ [7]. The two sets of rolls are born as stationary modes, with equal linear gain, and form an angle $\theta_{1,2} = \pm \arccos[2\pi/(q\Delta x)]$ with the direction of the wave front displacement. The roll amplitudes experience nonlinear competition, an effective diffusion arising from the curvature of the neutral surface and transport operated by the lateral wavefront shift. The asymptotic pattern observed after some time consists of an array of striped domains with irregular widths and parallel to the drift direction.

Model and simulations. To describe this process one may start from coupled amplitude equations for the envelope of the two degenerate roll systems. However, in order to bring

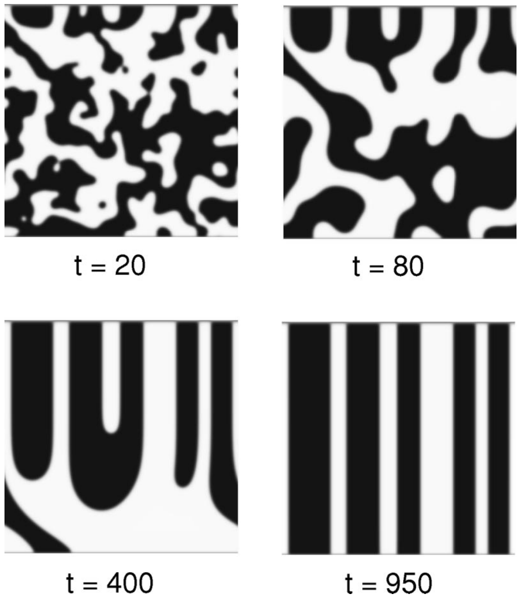

FIG. 1. Snapshots of the time evolution of the patterns obtained for $\lambda=1$, $D=0.2$, $v_g=0.2$ at four different times. The direction x of drift is downward. The size of the box is 100×100, spatial discretization $\Delta x=0.25$, time step $\Delta t=0.1$. The boundary conditions are $u=0$ for $x=0$, 100 and periodic in the y direction. The initial uncorrelated noise is uniformly distributed, $-10^{-2}<u<+10^{-2}$.

out the generic features more clearly, we here present a study of the minimal Ginzburg-Landau model for a real scalar field $u(\vec{r},t)$ $[\vec{r}\equiv(x,y)]$,

$$\partial_t u=\lambda u-u^3+D\nabla^2 u+v_g\partial_x u. \qquad (1)$$

Simulations with the coupled amplitude equations confirmed the robustness of our results. Equation (1) describes a bistable system with symmetry-degenerate states $u=\pm\sqrt{\lambda}$ and nonconserved order parameter (possibly an envelope). The group velocity $v_g(\geqslant 0)$ accounts for the drift along x. The linear growth rate λ and the diffusion coefficient D can be scaled away; however, in our presentation we explicitly keep them for the sake of clarity.

Starting from spatially distributed random initial conditions with zero average and rms $u_0\ll 1$, drift and diffusion act as follows. First, diffusion and linear growth together create a selective amplification of the long-wavelength variations (‘‘linear coarsening’’), yielding a random spatial distribution of the field. Then, the nonlinearity transforms that distribution into saturated patches of either phase separated by (comparatively) sharp interfaces moving under the influence of curvature and drift. The effect of the drift, together with nonperiodic boundary conditions, is to drive the 2D patches towards a quasi-1D pattern with a random distribution of stripes. Once the pattern has become 1D, a cut in the direction perpendicular to the drift shows a structure of kinks and antikinks. It is well established [10] that in an ideal system, this structure will eventually relax to one of the two phases. However, due to the very weak kink-antikink interactions, this takes an exponentially long time. Moreover, in real systems as well as in simulations this process is actually stopped at some stage due to the effect of pinning.

The scenario is illustrated in Fig. 1, where a numerical

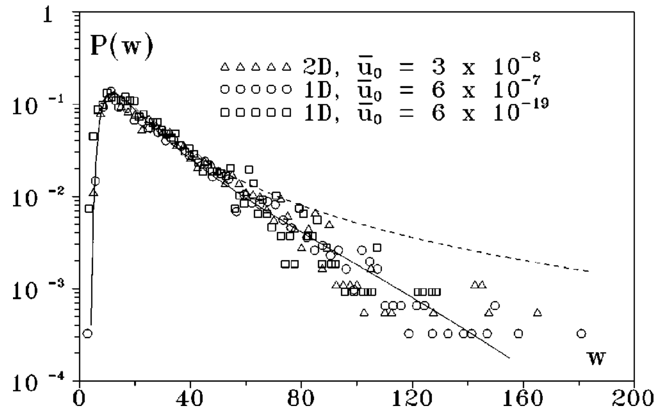

FIG. 2. Distribution of stripe widths as obtained from 200 simulations at velocity $v_g=0.2$ is shown on a log-linear scale (triangles). Same parameters as in Fig. 1, different noise realizations. Also shown are results from very long 1D systems (circles, squares). Dashed curve, distribution function $P_l(w)$ from Eq. (3); solid curve, corrected distribution $P_m(w)$ with $\alpha=0.015$.

simulation of Eq. (1) with nonperiodic boundary conditions without bias for one of the two stable phases is shown. With periodic boundary conditions in the direction of the drift, no stripes form because the drift term can be eliminated by a Galilean transformation.

Linear growth and saturation. To elucidate the quantitative aspects of this scenario, we consider Eq. (1) in a rectangular region $0<x<L_x$, $0<y<L_y$, with boundary conditions $u=0$ for $x=0$, L_x (note that Neumann conditions $\partial u/\partial x=0$ do not change qualitatively the pictures described here) and with periodic boundary conditions in the y direction ($L_x,L_y\gg\xi=\sqrt{D/\lambda}$). In the linear regime ($|u|\ll 1$), we expand u in terms of a set of modal solutions $e^{\sigma(q,p)t}F_{q,p}(x,y)$ that satisfy the boundary conditions. One finds $F_{q,p}=f_q(x)e^{ipy}$, where $f_q=e^{-x/l_D}\sin qx$, $p=2\pi n_y/L_y$, $n_y=0,\pm1,\ldots,$ $q=\pi n_x/L_x$, $n_x=1,2,\ldots$. Here, $l_D=2D/v_g$ is the ‘‘drift-versus-diffusion length’’ and

$$\sigma=\lambda'-D(q^2+p^2),\quad \lambda'=\lambda(1-v_g^2/v_{max}^2). \qquad (2)$$

The velocity $v_{max}=2\sqrt{\lambda D}$ separates the range of absolute ($v_g<v_{max}$) from that of convective instability ($v_g>v_{max}$) of the trivial solution $u=0$. In the convectively unstable range, the solutions $u=\pm\sqrt{\lambda}$ are, in the absence of permanent noise, swept out of the system by the drift term, restoring the trivial state. With our spatially distributed initial conditions the reduction of the growth rate by the drift [see Eq. (2)] is relevant only near the ‘‘inflow’’ boundary $x=0$ within a strip of width $\sim l_D$. Further downstream advection restores the unreduced growth. Our results for the case $v_g\sim v_{max}$ will demonstrate the influence of the reduction factor.

From Eq. (2) we see a selective amplification of modes near $q=p=0$ out of the broad band of modes initially excited. Thus, within the linear range, at time t, one expects a distribution of wave numbers p (irrespective of q) proportional to $P_l(p,t)=e^{-Dp^2t}$.

When u becomes of order 1 one enters the nonlinear regime. This occurs when the fastest growing mode $(q,p)=0$ reaches 1, i.e., when $\overline{u}_0e^{\lambda' t_l}\approx 1$, $\overline{u}_0=u_0/(N_xN_y)$, where $N_i=L_i/\Delta x$ are the number of lattice points in the two directions. The saturation processes actually represents an unsolved problem [11,12]. However, it rapidly leads to the regime of ''late time coarsening,'' where one has 2D patches with $u=\pm\sqrt{\lambda}$ separated by domain walls of width $\sim\xi=\sqrt{D/\lambda}$. The first snapshot in Fig. 1 corresponds to that stage.

The simplest theory for the distribution of domain widths w assumes that the distribution $P_l(p,t)$ of wave numbers in the y direction transforms at a time $t_c \approx t_l$ into a distribution of stripes of ''double width'' $w=2\pi/p$. The factor of 2π implies that the natural elementary unit comprises two (black-white) stripes. Thus, we have for the (unnormalized) distribution,

$$P_l(w)=|dp/dw|P_l(p)=(2\pi/w^2)e^{-(w_c/w)^2}, \quad (3)$$

with $w_c^2=(2\pi)^2 D t_c$, $t_c=\lambda'^{-1}\ln(1/\bar{u}_0)$.

Next we compare the distribution $P_l(w)$ with the results of simulations. In Fig. 2 the distribution of stripe widths obtained after a time $t=200$ is shown (triangles) for 100 simulations on a log-linear scale (same parameters as for Fig. 1). The dashed curve represents the distribution function (3) with $w_c=12.0$ calculated from the noise strength. Clearly the function (3) with adjusted prefactor (normalization) describes the simulations well except for large w (there the probability has already dropped by more than one order of magnitude). One may conclude that, at $v_g=0.2$, the width of the stripes is mainly determined at the moment when the nonlinear regime starts.

In order to examine the discrepancies at large widths we have redone the simulations for a larger system ($L_x=100$, $L_y=200$) without detecting a change (the triangles actually include these results). We also did analogous simulations on a very long 1D system ($L=L_y=2000$) with two very different noise strengths. Interestingly, after rescaling w_c according to the noise strength $\bar{u}_0$, the distribution of domain widths is essentially indistinguishable from the 2D stripe widths; see Fig. 2.

The simple distribution function $P_l(w)$ apparently overestimates the occurrence of wide domains. Indeed, the simple picture proposed above does not include processes where a wide domain splits into three by the insertion of a smaller domain. Although this appears to be an interesting problem worthy of further study we proceed phenomenologically by going over to $P_m(w)=P_l(w)\exp[-\alpha(w/w_c)^2]$; see Fig. 2 (solid line), where $\alpha=0.015$. We offer the following interpretation: after the long-wave components of u have saturated, the shorter-wave contributions still grow somewhat, thereby annihilating large domains (as mentioned before).

Thus, the final outcome is described rather well by a linear, and in fact even 1D, analysis of the problem, with a moderate nonlinear correction. This is to be expected for velocities such that the drift-versus-diffusion time $t_D=D/v_g^2$ is small compared to (or at most of the order of) t_c. If this is not the case 2D coarsening enters the problem. We now study these effects.

Nonlinear domain growth. In the sharp-interface regime the motion of the interface between different domains is driven by the local curvature κ according to the equation [13]

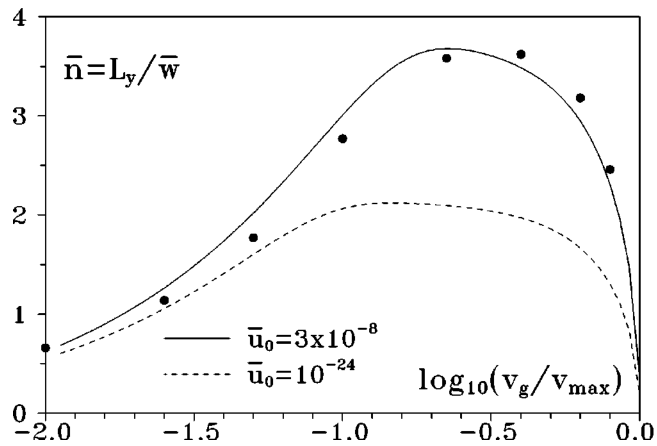

FIG. 3. Average number of double stripes $L/\bar{w}$ vs v_g/v_{max} from theory (solid and dashed curves). The symbols represent averaged results from various numbers of trial simulations for the larger noise amplitude (number of trials from left to right: 25,100,100,100,100,100). The discrepancies are presumably mainly due to fluctuations.

$$v_n=-D\kappa, \quad (4)$$

where v_n is the normal velocity of the interface in a frame moving with the drift velocity v_g. The coarsening process eliminates the small domains and tends to straighten out the interfaces; see Fig. 1, $t=80$. Solving the above equation one expects coarsening according to $R\propto\sqrt{Dt}$, where R is a measure of the domain size [11]. Note that this growth law is indeed similar to the ''linear coarsening'' affected by the selective amplification in the linear regime, as seen from the fall-off length w_c. The boundary conditions at $x=0$ in the sharp-interface regime orient the domain walls perpendicular to the boundaries.

In the simulations of Eq. (1), one often observes fingers of one phase traveling in the x direction with a constant velocity, preserving their shape; see Fig. 1. We found a family of exact solution of Eq. (4) that describes the fingers moving with velocity v_b,

$$Y=\pm 2\arctan(\sqrt{e^{X-X_0}-1})+Y_0, \quad (5)$$

where $x-v_b t=Xl'_D/4$ and $y=Yl'_D/4$, $l'_D=2D/v_b$. The width of the finger is 2π in the reduced units, or $b=\pi l'_D/2=\pi D/v_b$ in physical units. Thus, the velocity of the finger is a unique function of the width: $v_b=\pi D/b$, in the frame moving with the drift velocity. In this frame the fingers retract. For the finger to grow in the laboratory frame, v_g has to overcome v_b. Therefore, there is a minimum width $b_{min}=\pi D/v_g=\pi/2\ l_D$. Note that the solution (5) is different from the finger solution that occurs in crystal growth processes [14].

This amounts to saying that the tip curvature imposes a lower bound on the distribution of the stripe widths. The average finger width $\bar{w}$ is calculated from the distribution $P_m(w)$, limiting the integration to $w\geqslant w_{min}=2b_{min}$,

$$\bar{w}=\int_{2b_{min}}^{\infty} dw\, wP(w) \bigg/ \int_{2b_{min}}^{\infty} dw P(w). \quad (6)$$

In Fig. 3 we show the average number of double stripes $\overline{n} := L_y/\overline{w}$ for the noise level used in the simulations (and for much weaker noise). The increase of $\overline{w}$ for low drift velocities is a direct consequence of the cutoff introduced above. When $w_{min} << w_c$ (which means sufficiently high velocities), the lower cutoff becomes irrelevant. Then the integrals in Eq. (6) can be calculated analytically leading to $\overline{w} = 2\pi^{-1/2}\exp(2\alpha^{1/2})K_0(2\alpha^{1/2})w_c \approx 2.25w_c$. This approximates the results well in the region of the maximum and to the right of it.

Discussion and outlook. Coming back to the experiment that motivated our study, the observed transient dynamics and the asymptotic pattern agree qualitatively with the theoretical mechanism proposed here. Observation of the variation of the distribution of domain widths, as predicted by the model, would require either a variation of the diffusion coefficient not easily obtainable, or a variation of the drift velocity. However, the range of velocities where one has bistability is rather limited (see Ref. [7]). Thus, we propose to perform experiments in other optical or hydrodynamic bistable systems.

In the framework of our description, pinning of domain boundaries due to their interaction with the underlying pattern is not captured. It is described by generalized Swift-Hohenberg equations where it can stabilize the front [15]. Thus, it would be preferable to perform experiments in a system with homogeneous bistable states.

The finger solution presented for an interface moving under the influence of surface tension may be of general interest. It describes the asymptotics of a typical breakup scenario of lamellae in bistable systems. The generalization to a nonsymmetric situation where the flat interface moves with a velocity v_0 is also applicable to bistable and excitable chemical media [16].

Our analysis of the distribution of stripe widths shows that drift can serve as a tool to study diffusive coarsening processes. Selecting the drift velocity properly and introducing a time delay before application of the drift (this can certainly be done in simulations and in some experimental systems) one can get an extended snapshot of the state of the system near the inflow boundary at a chosen time. Thus, whereas fast drift that carries the system into the convectively unstable range can be used as a probe for subcritical noise, slow drift can probe some aspects of the supercritical, nonlinear regime.

Due to the generality of such a dynamical mechanism, it has to be expected in other systems where diffusion and drift act together. The case of a conserved order parameter, relating to spinodal decomposition, appears of interest. We expect similar qualitative features, although the distribution of stripe widths should be quite different.

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